ALGEBRA (OVERALL)

Problem 1.

(1) Prove that any finite field has order p^n for some prime p and integer n.
(2) State the law of quadratic reciprocity. For which odd prime p is -1 a square modulo p? For which odd prime p is 2 a square modulo p?
(3) Assuming that 691 is a prime, prove that 439 is not a square modulo 691.
(4) For which odd prime p does the polynomial $x^2 + 6x + 1$ have two roots in $\mathbb{Z}/p\mathbb{Z}$?

Problem 2. Let $G = \mathrm{GL}_2(\mathbb{F}_p)$

(1) Prove that the subgroup of upper triangular matrices with 1's on the diagonal is a Sylow p-subgroup of G.
(2) Compute the number of Sylow p-subgroups of G.